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[www.elsevier.com/locate/jnt](http://www.elsevier.com/locate/jnt)The Rogers–Ramanujan continued fraction and a new Eisenstein series identity<sup>☆</sup>Heng Huat Chan<sup>a</sup>, Song Heng Chan<sup>b</sup>, Zhi-Guo Liu<sup>c,\*</sup><sup>a</sup> Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 119260, Republic of Singapore<sup>b</sup> Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Republic of Singapore<sup>c</sup> Department of Mathematics, East China Normal University, 500 Dongchuan Road, Shanghai 200241, PR China

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## ABSTRACT

With two elementary trigonometric sums and the Jacobi theta function  $\theta_1$ , we provide a new proof of two Ramanujan's identities for the Rogers–Ramanujan continued fraction in his lost notebook. We further derive a new Eisenstein series identity associated with the Rogers–Ramanujan continued fraction.

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## 1. Introduction

Throughout this paper we will take  $q = e^{2\pi i\tau}$ , where  $\text{Im } \tau > 0$ . Let  $G(z)$  be defined as

$$G(z) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} z^n}{(1-q)(1-q^2)\cdots(1-q^n)}. \quad (1.1)$$

Then the celebrated Rogers–Ramanujan identities are given as follows [2, Chapter 7], [13, p. 290]:

$$G(1) = (q; q^5)_{\infty}^{-1} (q^4; q^5)_{\infty}^{-1} \quad \text{and} \quad G(q) = (q^2; q^5)_{\infty}^{-1} (q^3; q^5)_{\infty}^{-1}, \quad (1.2)$$

where

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$

The Rogers–Ramanujan continued fraction  $R(\tau)$  is defined by

$$R(\tau) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}}. \quad (1.3)$$

With the help of the Rogers–Ramanujan identities, both Rogers [22] and Ramanujan [20, Vol. II, Chapter 16, Section 15], [6, p. 30] prove that

$$R(\tau) = q^{1/5} \frac{G(q)}{G(1)} = q^{1/5} \frac{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}. \quad (1.4)$$

In his first two letters to G.H. Hardy [19, pp. xxvii, xxviii], [7, pp. 21–30, 53–62], S. Ramanujan made several claims about  $R(\tau)$ . In particular, in his first letter, he asserted that

$$R(i) = \sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}. \quad (1.5)$$

Moreover, in his notebooks [20] and “lost notebook” [21], Ramanujan recorded without proofs many results about  $R(\tau)$ , and many studies on these results have appeared, see for examples, [3, 4, 8, 9, 18, 24, 25].

In this paper, we establish several new identities associated with the Rogers–Ramanujan continued fraction  $R(\tau)$ . Using two elementary trigonometric sums and the Jacobi theta function  $\theta_1$ , we will derive the following theorem:

**Theorem 1.** Let  $|q| < 1$ ,  $\alpha = \frac{1-\sqrt{5}}{2}$  and  $\beta = \frac{1+\sqrt{5}}{2}$ . Then

$$\prod_{n=1}^{\infty} \frac{1}{(1 + \alpha q^n + q^{2n})} - \prod_{n=1}^{\infty} \frac{1}{(1 + \beta q^n + q^{2n})} = q^{1/12} \sqrt{\frac{5\eta(25\tau)}{\eta(5\tau)}} R(5\tau), \quad (1.6)$$

$$\beta \prod_{n=1}^{\infty} \frac{1}{(1 + \alpha q^n + q^{2n})} - \alpha \prod_{n=1}^{\infty} \frac{1}{(1 + \beta q^n + q^{2n})} = q^{1/12} \sqrt{\frac{5\eta(25\tau)}{\eta(5\tau)R(5\tau)}}, \quad (1.7)$$

where

$$\eta(\tau) = q^{1/24}(q; q)_{\infty}.$$

Identities (1.6) and (1.7) appear to be new. We observe that by replacing  $\tau$  by  $\tau/5$  in (1.6) and (1.7) and solving the resulting equations, we immediately deduce the following identities of S. Ramanujan [21, p. 206].

**Corollary 2.** Suppose  $|q| < 1$ . Let  $\alpha, \beta$  be given as in Theorem 1. Then

$$\prod_{n=1}^{\infty} \frac{1}{(1 + \alpha q^{n/5} + q^{2n/5})} = q^{1/60} \sqrt{\frac{\eta(5\tau)}{\eta(\tau)}} \left( \sqrt{\frac{1}{R(\tau)}} - \alpha \sqrt{R(\tau)} \right), \quad (1.8)$$

$$\prod_{n=1}^{\infty} \frac{1}{(1 + \beta q^{n/5} + q^{2n/5})} = q^{1/60} \sqrt{\frac{\eta(5\tau)}{\eta(\tau)}} \left( \sqrt{\frac{1}{R(\tau)}} - \beta \sqrt{R(\tau)} \right). \quad (1.9)$$

Corollary 2 was first discussed in a paper by K.G. Ramanathan [18, p. 220, (46) and (47)]. To prove Corollary 2, he stated without proofs the two identities [18, (50)]

$$g(q) - \alpha q^{1/5} h(q) = \prod_{n=1}^{\infty} (1 - q^{n/5})(1 + \beta q^{n/5} + q^{2n/5}), \quad (1.10)$$

$$g(q) - \beta q^{1/5} h(q) = \prod_{n=1}^{\infty} (1 - q^{n/5})(1 + \alpha q^{n/5} + q^{2n/5}), \quad (1.11)$$

where

$$h(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2+n)/2} \quad \text{and} \quad g(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2+3n)/2}.$$

The proofs of (1.10) and (1.11) were subsequently given by Berndt, Huang, Sohn, and Son [9] (see also [5, p. 21]). Our proofs of Corollary 2, which relies on (1.6) and (1.7), are different from those given in [9].

In [17], the third author established the following Eisenstein series identity involving  $R(\tau)$  by using (1.6).

**Theorem 3.** Let  $|q| < 1$ . Then

$$\begin{aligned} & \sum_{\substack{n=1 \\ n \equiv 2 \pmod{5}}}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} - \sum_{\substack{n=1 \\ n \equiv 3 \pmod{5}}}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} \\ &= \eta(\tau) \eta(25\tau) \sqrt{\frac{R(5\tau) \eta(25\tau)}{\eta(5\tau)}}. \end{aligned} \quad (1.12)$$

It turns out that (1.12) has the following companion:

**Theorem 4.** Let  $|q| < 1$ . Then

$$\begin{aligned} & \sum_{\substack{n=1 \\ n \equiv 1 \pmod{5}}}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} - \sum_{\substack{n=1 \\ n \equiv 4 \pmod{5}}}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} \\ &= \eta(\tau)\eta(25\tau) \sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}}. \end{aligned} \quad (1.13)$$

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1. Section 3 is devoted to the proof of Theorem 4.

## 2. The Jacobi theta function $\theta_1$ and the proof of Theorem 1

The Jacobi triple product identity [10, p. 10], [13, pp. 282–283] is perhaps the most important identity in the theory of elliptic theta functions, which states that

$$(q; q)_{\infty}(z; q)_{\infty}(q/z; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n, \quad z \neq 0. \quad (2.1)$$

The Jacobi triple product identity was established by Jacobi in his famous *Fundamenta Nova* [14], but, in fact, was first proved by Gauss [12]. There are now several elementary proofs of (2.1) which do not use elliptic functions, see for examples, [1, 11, 15].

If we multiply both sides of (2.1) with  $z^{-1/2}$ , then we can find that

$$(z^{-1/2} - z^{1/2})(q; q)_{\infty}(qz; q)_{\infty}(q/z; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^{n-1/2}.$$

Replacing  $z$  by  $e^{2iz}$  and  $e^{-2iz}$  respectively in the equation above, we conclude that

$$\begin{aligned} 2i(\sin z)(q; q)_{\infty}(qe^{2iz}; q)_{\infty}(qe^{-2iz}; q)_{\infty} &= - \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} e^{(2n-1)iz}, \\ 2i(\sin z)(q; q)_{\infty}(qe^{2iz}; q)_{\infty}(qe^{-2iz}; q)_{\infty} &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} e^{-(2n-1)iz}. \end{aligned}$$

Making the index change  $n \rightarrow n+1$  in the right-hand sides of the two equations above, adding the two resulting equations together, we arrive at

$$\begin{aligned} 2(\sin z)(q; q)_{\infty}(qe^{2iz}; q)_{\infty}(qe^{-2iz}; q)_{\infty} &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z \\ &= 2 \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z. \end{aligned} \quad (2.2)$$

The well-known Jacobi theta function  $\theta_1$  is defined as follows

$$\theta_1(z|\tau) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/8} \sin(2n+1)z = 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z. \quad (2.3)$$

Combining (2.2) and (2.3), we immediately find the infinite product representation of  $\theta_1$ :

$$\theta_1(z|\tau) = 2q^{1/8} (\sin z)(q; q)_{\infty} (qe^{2iz}; q)_{\infty} (qe^{-2iz}; q)_{\infty}. \quad (2.4)$$

Let  $z = \frac{\pi}{5}$  and  $z = \frac{2\pi}{5}$  respectively in (2.4). Then we have

$$\theta_1\left(\frac{\pi}{5} \middle| \tau\right) = 2q^{1/8} \left(\sin \frac{\pi}{5}\right) (q; q)_{\infty} (qe^{\frac{2\pi i}{5}}; q)_{\infty} (qe^{-\frac{2\pi i}{5}}; q)_{\infty}, \quad (2.5)$$

and

$$\theta_1\left(\frac{2\pi}{5} \middle| \tau\right) = 2q^{1/8} \left(\sin \frac{2\pi}{5}\right) (q; q)_{\infty} (qe^{\frac{4\pi i}{5}}; q)_{\infty} (qe^{-\frac{4\pi i}{5}}; q)_{\infty}. \quad (2.6)$$

Multiplying (2.5) and (2.6) together, and using the elementary identities

$$\sin \frac{\pi}{5} \sin \frac{2\pi}{5} = \frac{\sqrt{5}}{4}$$

and

$$(1 - xe^{\frac{2\pi i}{5}})(1 - xe^{-\frac{2\pi i}{5}})(1 - xe^{\frac{4\pi i}{5}})(1 - xe^{-\frac{4\pi i}{5}}) = 1 - x^5$$

in the resulting equation, and simplifying, we deduce the following useful identity:

$$\theta_1\left(\frac{\pi}{5} \middle| \tau\right) \theta_1\left(\frac{2\pi}{5} \middle| \tau\right) = \sqrt{5} \eta(\tau) \eta(5\tau). \quad (2.7)$$

We now give a proof of Theorem 1.

**Proof.** It is easily seen that we may rewrite (2.5) and (2.6) as follows

$$\theta_1\left(\frac{\pi}{5} \middle| \tau\right) = 2q^{1/12} \left(\sin \frac{\pi}{5}\right) \eta(\tau) \prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}), \quad (2.8)$$

and

$$\theta_1\left(\frac{2\pi}{5} \middle| \tau\right) = 2q^{1/12} \left(\sin \frac{2\pi}{5}\right) \eta(\tau) \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n}), \quad (2.9)$$

where

$$\alpha = -2 \cos \frac{2\pi}{5} = \frac{1 - \sqrt{5}}{2} \quad \text{and} \quad \beta = -2 \cos \frac{4\pi}{5} = 2 \cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{2}.$$

It follows that

$$\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}) = q^{-1/12} \frac{1}{\eta(\tau)} \frac{\theta_1(\frac{\pi}{5}|\tau)}{2 \sin \frac{\pi}{5}} \quad (2.10)$$

and

$$\prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n}) = q^{-1/12} \frac{1}{\eta(\tau)} \frac{\theta_1(\frac{2\pi}{5}|\tau)}{2 \sin \frac{2\pi}{5}}. \quad (2.11)$$

Multiplying the two equations above together and then using (2.7), we deduce that

$$\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n})(1 + \beta q^n + q^{2n}) = q^{-1/6} \frac{\eta(5\tau)}{\eta(\tau)}. \quad (2.12)$$

We observe that by multiplying (1.8) and (1.9) and using (2.12), we obtain the following identity of Ramanujan (see [6, pp. 265–267], [16]):

$$\frac{1}{R(\tau)} - 1 - R(\tau) = \frac{\eta(\frac{\tau}{5})}{\eta(5\tau)}. \quad (2.13)$$

If we subtract (2.11) from (2.10), then we find that

$$\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}) - \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n}) = q^{-1/12} \frac{1}{\eta(\tau)} \left( \frac{\theta_1(\frac{\pi}{5}|\tau)}{2 \sin \frac{\pi}{5}} - \frac{\theta_1(\frac{2\pi}{5}|\tau)}{2 \sin \frac{2\pi}{5}} \right). \quad (2.14)$$

Using (2.3), we simplify the right-hand side of the equation above as

$$q^{-1/12} \frac{1}{\eta(\tau)} \sum_{n=0}^{\infty} (-1)^n A(n) q^{(2n+1)^2/8}, \quad (2.15)$$

where

$$A(n) = \frac{\sin(2n+1)\frac{\pi}{5}}{\sin \frac{\pi}{5}} - \frac{\sin(2n+1)\frac{2\pi}{5}}{\sin \frac{2\pi}{5}}. \quad (2.16)$$

Direct computations show that

$$A(5n+1) = \sqrt{5}, \quad A(5n+3) = -\sqrt{5}, \quad A(5n) = A(5n+2) = A(5n+4) = 0.$$

Hence,

$$\sum_{n=0}^{\infty} (-1)^n A(n) q^{(2n+1)^2/8} = -\sqrt{5} \sum_{n=-\infty}^{\infty} (-1)^n q^{(10n-3)^2/8},$$

and the Jacobi triple product identity tells us that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(10n-3)^2/8} = q^{9/8} (q^5; q^{25})_{\infty} (q^{20}; q^{25})_{\infty} (q^{25}; q^{25})_{\infty}.$$

Thus, we have

$$\sum_{n=0}^{\infty} (-1)^n A(n) q^{(2n+1)^2/8} = -\sqrt{5} q^{9/8} (q^5; q^{25})_{\infty} (q^{20}; q^{25})_{\infty} (q^{25}; q^{25})_{\infty}. \quad (2.17)$$

Combining (2.14)–(2.17), we conclude that

$$\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}) - \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n}) = -\sqrt{5} \frac{\eta(25\tau)}{\eta(\tau)} (q^5; q^{25})_{\infty} (q^{20}; q^{25})_{\infty}. \quad (2.18)$$

Dividing both sides of the equation above by  $\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n})(1 + \beta q^n + q^{2n})$ , we deduce that

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n})^{-1} - \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n})^{-1} \\ &= \sqrt{5} \frac{\eta(25\tau)}{\eta(\tau)} \frac{(q^5; q^{25})_{\infty} (q^{20}; q^{25})_{\infty}}{\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n})(1 + \beta q^n + q^{2n})}. \end{aligned}$$

Substituting (2.12) into the right-hand side of the equation above, we find that

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n})^{-1} - \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n})^{-1} \\ &= \sqrt{5} q^{1/6} \frac{\eta(25\tau)}{\eta(5\tau)} (q^5; q^{25})_{\infty} (q^{20}; q^{25})_{\infty} \\ &= q^{1/12} \sqrt{\frac{5\eta(25\tau)}{\eta(5\tau)} R(5\tau)}, \end{aligned}$$

which is (1.6).

Next, we prove (1.7). Multiplying (2.10) by  $\cos \frac{2\pi}{5}$  and (2.11) by  $\cos \frac{\pi}{5}$  and then adding the two resulting equations together, we find that

$$\begin{aligned} & 2 \cos \frac{2\pi}{5} \prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}) + 2 \cos \frac{\pi}{5} \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n}) \\ &= q^{-1/12} \frac{1}{\eta(\tau)} \left( \frac{\cos \frac{2\pi}{5}}{\sin \frac{\pi}{5}} \theta_1 \left( \frac{\pi}{5} \middle| \tau \right) + \frac{\cos \frac{\pi}{5}}{\sin \frac{2\pi}{5}} \theta_1 \left( \frac{2\pi}{5} \middle| \tau \right) \right). \end{aligned} \quad (2.19)$$

Using (2.3), we rewrite (2.19) as

$$q^{-1/12} \frac{1}{\eta(\tau)} \sum_{n=0}^{\infty} (-1)^n B(n) q^{(2n+1)^2/8}, \quad (2.20)$$

where

$$B(n) = 2 \frac{\cos \frac{2\pi}{5}}{\sin \frac{\pi}{5}} \sin(2n+1) \frac{\pi}{5} + 2 \frac{\cos \frac{\pi}{5}}{\sin \frac{2\pi}{5}} \sin(2n+1) \frac{2\pi}{5}. \quad (2.21)$$

Direct computations show that

$$B(5n) = \sqrt{5}, \quad B(5n+4) = -\sqrt{5}, \quad B(5n+1) = B(5n+2) = B(5n+3) = 0.$$

Thus, by (2.1), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n B(n) q^{(2n+1)^2/8} &= \sqrt{5} \sum_{n=-\infty}^{\infty} (-1)^n q^{(10n+1)^2/8} \\ &= \sqrt{5} q^{1/8} (q^{10}; q^{25})_{\infty} (q^{15}; q^{25})_{\infty} (q^{25}; q^{25})_{\infty}. \end{aligned}$$

It follows that

$$\begin{aligned} 2 \cos \frac{2\pi}{5} \prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}) + 2 \cos \frac{\pi}{5} \prod_{n=1}^{\infty} (1 + \beta q^n + q^{2n}) \\ = \sqrt{5} q^{-1} \frac{\eta(25\tau)}{\eta(\tau)} (q^{10}; q^{25})_{\infty} (q^{15}; q^{25})_{\infty}. \end{aligned} \quad (2.22)$$

Proceeding through the same steps as in the proof of (1.6), we divide both sides of the equation above by

$$\prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n})(1 + \beta q^n + q^{2n}) = q^{-1/6} \frac{\eta(5\tau)}{\eta(\tau)},$$

and arrive at (1.7). Thus we complete the proof of Theorem 1.  $\square$

### 3. A new Eisenstein series identity associated with $R(\tau)$

In this section, we will prove Theorem 4.

**Proof of Theorem 4.** We first recall the identity [17, p. 296, Eq. (5.8)]:

$$\sum_{n=1}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} \sin 2n\tau = \frac{\eta^2(\tau)\theta_1(z|5\tau)\theta_1(2z|5\tau)}{2\eta(5\tau)\theta_1(z|\tau)}. \quad (3.1)$$

Taking  $z = \frac{\pi}{5}$  and then multiplying both sides by  $2\beta = 1 + \sqrt{5} = 2 \cos \frac{\pi}{5}$  we find that

$$\sum_{n=1}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} \sin \frac{2\pi}{5} \sin \frac{2n\pi}{5} = \frac{\sqrt{5}}{4} q^{-1/12} \eta(\tau) \eta(25\tau) \prod_{n=1}^{\infty} \frac{\beta}{1 + \alpha q^n + q^{2n}}. \quad (3.2)$$

In the same way by taking  $z = \frac{2\pi}{5}$  in (3.1) and then multiplying both sides by

$$2\alpha = 1 - \sqrt{5} = -2 \cos \frac{2\pi}{5},$$



we obtain

$$-\sum_{n=1}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} \sin \frac{4\pi}{5} \sin \frac{4n\pi}{5} = \frac{\sqrt{5}}{4} q^{-1/12} \eta(\tau) \eta(25\tau) \prod_{n=1}^{\infty} \frac{\alpha}{1 + \beta q^n + q^{2n}}. \quad (3.3)$$

Subtracting (3.3) from (3.2), we deduce that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} S(n) \\ &= \frac{\sqrt{5}}{4} q^{-1/12} \eta(\tau) \eta(25\tau) \left( \prod_{n=1}^{\infty} \frac{\beta}{1 + \alpha q^n + q^{2n}} - \prod_{n=1}^{\infty} \frac{\alpha}{1 + \beta q^n + q^{2n}} \right), \end{aligned} \quad (3.4)$$

where

$$S(n) = \sin \frac{2\pi}{5} \sin \frac{2n\pi}{5} + \sin \frac{4\pi}{5} \sin \frac{4n\pi}{5}. \quad (3.5)$$

Substituting (1.7) into the right-hand side of (3.4), we deduce that

$$\sum_{n=1}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} S(n) = \frac{5}{4} \eta(\tau) \eta(25\tau) \sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}}. \quad (3.6)$$

By direct computation, we have

$$\begin{aligned} S(5n) = S(5n+2) = S(5n+3) &= 0, & S(5n+1) &= \sin^2 \frac{\pi}{5} + \sin^2 \frac{2\pi}{5} = \frac{5}{4}, \\ S(5n+4) &= -\sin^2 \frac{\pi}{5} - \sin^2 \frac{2\pi}{5} = -\frac{5}{4}. \end{aligned} \quad (3.7)$$

Combining (3.6) and (3.7), we complete the proof of (1.13).  $\square$

#### 4. Some concluding remarks

4.1. Let  $G$  denotes the multiplicative group

$$(\mathbf{Z}/25\mathbf{Z})^* := \{[k]_{25} \mid \gcd(k, 5) = 1, 1 \leq k \leq 25\}.$$

Note that  $G$  is cyclic and generated by  $[2]_{25}$ . Let  $\tilde{\chi}$  be defined by

$$\tilde{\chi}([2]_{25}) = i$$

and extend it to a Dirichlet character on  $\mathbf{Z}$  by

$$\chi(n) = \begin{cases} \tilde{\chi}([n]_{25}) & \text{if } \gcd(n, 5) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then (1.12) and (1.13) can be rewritten as

$$\sum_{n=1}^{\infty} \chi(n) \frac{q^n - q^{2n} - q^{3n} + q^{4n}}{1 - q^{5n}} = \frac{\eta(\tau) \eta^{3/2}(25\tau)}{\eta^{1/2}(5\tau)} \left( \sqrt{\frac{1}{R(5\tau)}} + i\sqrt{R(5\tau)} \right).$$

Interpreting identities such as (1.12) and (1.13) with the help of group characters will certainly assist us in understanding their existence.

4.2. If we set  $z = \frac{\pi}{3}$  in (3.1) and use  $\theta_1(\frac{\pi}{3} | \tau) = \sqrt{3}\eta(3\tau)$  in the resulting equation, we find that

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} = \frac{\eta^2(\tau) \eta^2(15\tau)}{\eta(3\tau) \eta(5\tau)}. \quad (4.1)$$

We may rewrite the left-hand side of (4.1) as

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{3}\right) \left(\frac{n}{5}\right) q^{mn}. \quad (4.2)$$

Interchanging summation, we deduce that

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n - q^{2n}}{1 - q^{3n}} = \frac{\eta^2(\tau) \eta^2(15\tau)}{\eta(3\tau) \eta(5\tau)}. \quad (4.3)$$

The expression (4.2) is an example of  $q$ -series associated with genus characters of the imaginary quadratic field  $\mathbf{Q}(\sqrt{-15})$ . Series associated with genus characters are recently used by P.C. Toh [23] to derive explicit representations of certain theta series associated with positive definite quadratic forms in terms of Lambert series. For example, using the ideas in [23], one can show that

$$\frac{1}{2} \left( \sum_{m,n \in \mathbf{Z}} q^{m^2 + mn + 4n^2} - \sum_{m,n \in \mathbf{Z}} q^{3m^2 + 3mn + 2n^2} \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{3}\right) \left(\frac{n}{5}\right) q^{mn}. \quad (4.4)$$

Our identity (4.1) shows that indeed the left-hand side of (4.4) has a product representation, namely,

$$\frac{1}{2} \left( \sum_{m,n \in \mathbf{Z}} q^{m^2 + mn + 4n^2} - \sum_{m,n \in \mathbf{Z}} q^{3m^2 + 3mn + 2n^2} \right) = \frac{\eta^2(\tau) \eta^2(15\tau)}{\eta(3\tau) \eta(5\tau)}.$$

4.3. In this subsection, we present a different proof of Theorem 4 using (2.13) and the identity

$$\frac{(az, q/(az), q, q)_{\infty}}{(a, q/a, z, q/z; q)_{\infty}} = \sum_{n=-\infty}^{\infty} \frac{z^n}{1 - aq^n} \quad (|q| < |z| < 1), \quad (4.5)$$

where  $(a_1, a_2, \dots, a_n; q)_{\infty} := (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_n; q)_{\infty}$ . Theorem 3 can also be proved similarly.

The identity (4.5) is often referred to as a corollary of Ramanujan's  ${}_1\psi_1$  summation formula [6, Entry 17 p. 32] (and can be obtained by setting  $b = aq$  in [6, Entry 17 p. 32]).

**Second proof of Theorem 4.** Noting that

$$\sum_{n=0}^{\infty} \frac{x^n}{1-yq^n} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^n y^m q^{mn} = \sum_{n=0}^{\infty} \frac{y^n}{1-xq^n} \quad (|x|, |y| < 1),$$

the left-hand side of (1.13) simplifies to become,

$$\begin{aligned} & \sum_{\substack{n=1 \\ n \equiv 1 \pmod{5}}}^{\infty} \frac{(q^n - q^{2n} - q^{3n})}{1 - q^{5n}} - \sum_{\substack{n=1 \\ n \equiv 4 \pmod{5}}}^{\infty} \frac{(-q^{2n} - q^{3n} + q^{4n})}{1 - q^{5n}} \\ &= \sum_{\substack{n=1 \\ n \equiv 1 \pmod{5}}}^{\infty} \frac{(q^n - q^{2n} - q^{3n})}{1 - q^{5n}} - \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{5}}}^{-1} \frac{(-q^{-2n} - q^{-3n} + q^{-4n})}{1 - q^{-5n}} \\ &= \sum_{n=-\infty}^{\infty} \frac{q^{5n+1}}{1 - q^{25n+5}} - \sum_{n=-\infty}^{\infty} \frac{q^{10n+2}}{1 - q^{25n+5}} - \sum_{n=-\infty}^{\infty} \frac{q^{15n+3}}{1 - q^{25n+5}} \\ &= q \frac{(q^{10}, q^{15}, q^{25}, q^{25}; q^{25})_{\infty}}{(q^5, q^{20}; q^{25})_{\infty}^2} - q^2 \frac{(q^{10}, q^{15}, q^{25}, q^{25}; q^{25})_{\infty}}{(q^5, q^{10}, q^{15}, q^{20}; q^{25})_{\infty}} - q^3 \frac{(q^5, q^{20}, q^{25}, q^{25}; q^{25})_{\infty}}{(q^5, q^{10}, q^{15}, q^{20}; q^{25})_{\infty}} \\ &= q \frac{(q^{10}, q^{15}, q^{25}, q^{25}; q^{25})_{\infty}}{(q^5, q^{20}; q^{25})_{\infty}^2} - q^2 \frac{(q^{25}; q^{25})_{\infty}^2}{(q^5, q^{20}; q^{25})_{\infty}} - q^3 \frac{(q^{25}; q^{25})_{\infty}^2}{(q^{10}, q^{15}; q^{25})_{\infty}} \\ &= q^2 \frac{(q^{25}; q^{25})_{\infty}^2}{(q^5, q^{20}; q^{25})_{\infty}} \left\{ \frac{1}{R(5\tau)} - 1 - R(5\tau) \right\} \\ &= \eta^2(25\tau) \sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}} \frac{\eta(\tau)}{\eta(25\tau)} \\ &= \eta(\tau)\eta(25\tau) \sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}}, \end{aligned}$$

where we applied identity (4.5) three times in the third equality and (2.13) in the penultimate equality.  $\square$

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